The Specific Case: $|\mathfrak{R}| > |\mathbf{N}|$

1. Assumption for reductio: $\neg|\mathfrak{R}| > |\mathbf{N}|$.
2. Hence (by 1) $|\mathfrak{R}| \leq |\mathbf{N}|$.
3. Useful Lemma: for any sets **A** and **B**, if there exists a 1-to-1 function on **A** and into **B**, then $|\mathbf{A}| \leq |\mathbf{B}|$.
4. A 1-to-1 function that is on **N** and into $\mathfrak{R}$ is the identity function, which relates each $\mathbf{x} \in \mathbf{N}$ to itself in $\mathfrak{R}$. (Note that $\mathbf{N} \subset \mathfrak{R}$.).
5. Thus (by 3 and 4) $|\mathbf{N}| \leq |\mathfrak{R}|$.
6. If $|\mathfrak{R}| \leq |\mathbf{N}|$ and $|\mathbf{N}| \leq |\mathfrak{R}|$, then $|\mathbf{N}|=|\mathfrak{R}|$.
7. By definition of cardinality, since $|\mathbf{N}|=|\mathfrak{R}|$, there exists a function **f** that is on **N** and onto $\mathfrak{R}$ that is 1-to-1.
8. Using **f**, we can identify a number we will call **k** (for krazy), where $0 \leq \mathbf{k} \leq 1$. The $\mathbf{n}^{th}$ decimal of **k** is the $\mathbf{n}^{th}$ decimal of **f(n)**, plus 1, modulo 10.
9. Note that **k** is real, that is, $\mathbf{k} \in \mathfrak{R}$.
10. We assumed (by 7) that every real number is in the range of our function **f**. That means, for some **n**, **f(n)=k**.
11. But for any **n**, **f(n)** differs from **k** at least at the $\mathbf{n}^{th}$ decimal place (by 7).
12. Thus (by 11) it is not the case that for some **n**, **f(n)=k**.
13. We have a contradiction (by 10 and 12).
14. We conclude that the source of the contradiction is line 1. Hence $|\mathfrak{R}| > |\mathbf{N}|$.

NOTE: sometimes students get confused and assume that we've shown that $|\mathfrak{R}| > |\mathbf{N}|$ by showing that there is a thing **k** in $\mathfrak{R}$ that's not in **N**, as if we were proving that $\mathfrak{R}$ is bigger than **N** by one extra number. This is not what we showed, and that would not work. Adding one element to an infinite set does not make it larger. Rather, what we did was contradict the claim that there exists a function that could show they have the same cardinality.

The General Case: Cantor's Theorem. For any set S, $|\wp(\mathbf{S})| > |\mathbf{S}|$.

1. Assumption for reductio: for arbitrary set **S**, $\neg|\wp(\mathbf{S})| > |\mathbf{S}|$.
2. Hence (by 1) $|\wp(\mathbf{S})| \leq |\mathbf{S}|$.
3. Useful Lemma: for any sets **A** and **B**, if there exists a 1-to-1 function on **A** and into **B**, then $|\mathbf{A}| \leq |\mathbf{B}|$.
4. A 1-to-1 function that is on **S** and into $\wp(\mathbf{S})$ is the function that relates each $\mathbf{x} \in \mathbf{S}$ to the set $\{\mathbf{x}\}$. (Note: by definition of powerset, for any $\mathbf{x} \in \mathbf{S}$, $\{\mathbf{x}\} \in \wp(\mathbf{S})$.)
5. Thus (by 3 and 4) $|\mathbf{S}| \leq |\wp(\mathbf{S})|$.
6. If $|\wp(\mathbf{S})| \leq |\mathbf{S}|$ and $|\mathbf{S}| \leq |\wp(\mathbf{S})|$, then $|\wp(\mathbf{S})| = |\mathbf{S}|$.
7. By definition of cardinality, since $|\wp(\mathbf{S})| = |\mathbf{S}|$, there exists a function **f** on **S** and onto $\wp(\mathbf{S})$ that is 1-to-1.
8. Consider the set **K** (for Krazy set), where $\mathbf{K} = \{\mathbf{x} \mid \mathbf{x} \notin \mathbf{f(x)}\}$.
9. Note that $\mathbf{K} \in \wp(\mathbf{S})$.
10. We assumed (by 7) that every member of $\wp(\mathbf{S})$ is in the range of our function **f**. That means, for some $\mathbf{x} \in \mathbf{S}$, **f(x)=K**.
11. But if for some $\mathbf{x} \in \mathbf{S}$, **f(x)=K**, there will be a contradiction: either (a) If $\mathbf{x} \in \mathbf{K}$, since **f(x)=K**, then $\mathbf{x} \in \mathbf{f(x)}$, contradicting the definition that specifies $\mathbf{x} \notin \mathbf{f(x)}$; or (b) if $\mathbf{x} \notin \mathbf{K}$ since **f(x)=K** it follows that $\mathbf{x} \notin \mathbf{f(x)}$; but then by definition of **K**, $\mathbf{x} \in \mathbf{K}$.
12. Thus (by 11) it is not the case that for some $\mathbf{x} \in \mathbf{S}$, **f(x)=K**.
13. We have a contradiction (by 10 and 12).
14. We conclude that the source of the contradiction is line 1. Hence for any set **S**, $|\wp(\mathbf{S})| > |\mathbf{S}|$.