<u>Some Notes on Set Theory</u> <u>(2021)</u>
As a starting place (which we will later modify), we can say a set is any collection of things such that we can clearly identify for each object whether it is, or is not, in that set.

We will write **{...}** for a set when we want to show its contents, and **A**, **B**, ... for sets when we are dealing with them more generally. We write $\mathbf{x \in A}$ to mean that **x** is an element or member in the set **A**. Sets can be members of sets. We write $\mathbf{\{x \mid P(x)\}}$ to mean the set of elements that have the property **P**.

Sets are defined by their contents, so two sets are the same set if they have the same contents.

$$\mathbf{A = B} \text{ if and only if } \mathbf{\forall x(x \in A \leftrightarrow x \in B)}$$

If all the contents of a set **A** are in other set **B**, we say **A** is a subset of **B**.

$$\mathbf{A \subseteq B} \text{ if and only if } \mathbf{\forall x(x \in A \rightarrow x \in B)}$$

A proper subset is a subset that is not identical (that means **B** has something not in **A**, in this case):

$$\mathbf{A \subset B} \text{ if and only if } \mathbf{(A \subseteq B \wedge \neg A=B)}$$

Note that there is an empty set,

$$\{\}$$

which we can also write **Ø**. We can say of the empty set that $\mathbf{\forall x \neg (x \in \emptyset)}$.

The power set operation gives us the set of all subsets of a set.

$$\wp(\mathbf{A}) = \mathbf{B} \text{ if and only if } \mathbf{\forall x(x \subseteq A \rightarrow x \in B)}$$

For any set **S**, if **S** has n members, then the powerset $\wp(\mathbf{S})$ has 2^n members.

The ordinal size of a set is determined by comparing its contents with some kind of ordering (such as the natural numbers).

Cardinal size is determined by finding a one-to-one correspondence with the elements of the set. Two sets have the same cardinal size (we say, they have the same cardinality) if there is some way to show there exists a one-to-one correspondence between all the elements of one set and all the elements of the other. For the cardinality of some set **A**, we can write

$$|\mathbf{A}|$$

and thus there is a one-to-one correspondence to be found between all the elements of **A** and all the elements of **B**, if and only if

$$|\mathbf{A}| = |\mathbf{B}|$$

Note that if $\mathbf{A \subseteq B}$ then $|\mathbf{A}| \leq |\mathbf{B}|$.

The union of two sets is a set that contains every element of either set.

$\mathbf{A \cup B}$ is defined as satisfying $\mathbf{\forall x((x{\in}A \lor x{\in}B) \leftrightarrow x \in A \cup B)}$

The intersection of two sets is a set that contains every element in both the sets.

$\mathbf{A \cap B}$ is defined as satisfying $\mathbf{\forall x((x{\in}A \wedge x{\in}B) \leftrightarrow x \in A \cap B)}$

Normal sets are not ordered. Thus

$\mathbf{\{1, 2, 3\} = \{3, 2, 1\}}$

But an ordered set is a set in which the order does matter. We can indicate an ordered set using parentheses or using angle brackets, instead of curly brackets. Thus:

$\mathbf{(1, 2, 3) = (1, 2, 3)}$

but

$\mathbf{(1, 2, 3) \neq (3, 2, 1)}$

The Cartesian product of two sets is the set of all ordered pairs that can be formed with the members of both sets. We write **A x B** for the Cartesian product of **A** and **B**. This is the set of all the ordered pairs of members of **A** and **B**: $\mathbf{A \times B = \{(x, y) \mid x{\in}A \wedge x{\in}B\}}$.

A function **F** can be thought of as a relation between two sets. One set is called the *domain*, and the other is called the *range*. Suppose **A** is the *domain* and **B** is the *range* of a function, then (if we let **a** be a member of **A** and **b** and **c** be members of **B**, so by writing **Fab** I mean that function **F** relates **a** from its domain to **b** in its range):

If **F** is a function from **A** into **B**, then if **Fab** and **Fac** then **b=c**

We also say a function **F** is

- *from* a set **A** if its domain is a subset (not necessarily proper) of **A**
- *on* a set **A** if its domain is **A**
- *into* a set **B** if its range is a subset (not necessarily proper) of **B**
- *onto* a set **B** if its range is **B**

If a function **F** is such that

If **Fab** and **Fcb**, then **a=c**

then **F** is a 1-to-1 function. The idea of being 1-to-1 is that **F** is a function that if reversed would be a function also. If there is a function that is on **A**, and onto **B**, and is 1-to-1, then $\mathbf{|A| = |B|}$.

Another useful principle is the following: for any sets **A** and **B**, if there exists a 1-to-1 function on **A** and into **B**, then $\mathbf{|A| \leq |B|}$.