PHL310 Valid Reasoning II **NAME: __________________**
Quiz 3

A. Answer 4 of the following 7 questions (5 points each)**.**

1. How do we define cardinality? How does this differ from, for example, the classical idea that a proper subsets of set A should be smaller than set A?
2. Suppose we make a standard model for our arithmetic (axioms L1-L5 plus the Peano axioms). The domain of discourse of our model is the natural numbers. Our arithmetic contains an arity two predicate, F, which we typically write as "=". In a typical extensional semantics model, what is the interpretation of this predicate?
3. What is Cantor's Theorem, general form? What does Cantor's Theorem tell us about the maximal size of sets?
4. What does it mean to say that our propositional logic is complete?
5. What is the difference between a de dicto and de re reading of the modal necessity operator?
6. What is Cantor's Antinomy? How do we attempt to avoid it in the ZF set theory?
7. How might modal logic help us to capture some unusual features of a natural language (such as the sentence, "If Josh is a fish, then he's a mammal.")? Roughly, how do we construct a model for a modal logic, and how might it help us interpret these cases that are hard to capture in PL or FOL?

B. Complete 4 of the following 7 problems (10 points each)**.**

1. Using our axiomatic propositional logic (axioms L1-L3, modus ponens, and <u>without</u> the deduction theorem), prove:
   $$\{(A \rightarrow (B \rightarrow C)), B\} \vdash (A \rightarrow C)$$
2. Using our axiomatic quantified logic (axioms L1-L5, modus ponens, universal generalization, and deduction theorem), prove:
   $$(\forall x(Bx \rightarrow Cx) \rightarrow \forall x(\neg Cx \rightarrow \neg Bx))$$
3. Using any of our tools, prove: $(A \subseteq B \leftrightarrow A \cap B = A)$
4. Prove Cantor's Theorem, either the specific or the general form.
5. Using our axiomatic quantified logic and Peano Arithmetic (that means, using at most axioms L1-L5, and axioms A1-A9, and the rules of these systems), prove that $(1 \times (1+1)) = 2$. That is, prove that:
   $$0' * (0' + 0') = 0''$$
6. Prove the deduction theorem for propositional logic.
7. Using S5, prove:
   $$(\Diamond\Box B \leftrightarrow \Box B).$$

AXIOMS

PL (Propositional Logic)

(L1) $(B \rightarrow (C \rightarrow B)$
(L2) $((B \rightarrow (C \rightarrow D)) \rightarrow ((B \rightarrow C) \rightarrow (B \rightarrow D)))$
(L3) $((\neg C \rightarrow \neg B) \rightarrow ((\neg C \rightarrow B) \rightarrow C))$

And modus ponens.

FOL (First Order Logic)

PL and:

(L4) $\forall x_i(Bx_i) \rightarrow B(t)$
(L5) $\forall x_i(B \rightarrow C) \rightarrow (B \rightarrow \forall x_i C)$ if x_i does not appear free in B.

And generalization.

M (aka T)

PL and:

(m1): $(\Box B \rightarrow B)$
(m2): $(\Box(B \rightarrow C) \rightarrow (\Box B \rightarrow \Box C))$

And as an additional rule:

Necessitation: If $\varnothing \vdash B$ then $\varnothing \vdash \Box B$

Brouwer

PL and **M** and:

(m3): $(B \rightarrow \Box\Diamond B)$

S4

PL and **M** and:

(m4): $(\Box B \rightarrow \Box\Box B)$

S5

PL and **M** and:

(m5): $(\Diamond B \rightarrow \Box\Diamond B)$

ZFC axioms

ZFC is FOL with a single predicate for membership, along with the following axioms:

Existence: There exists a set that has no elements.

Extensionality: If every element of X is an element of Y, and every element of Y is an element of X, then X=Y.

Schema of comprehension: Let P(x) be a property of x. For any A, there is a B such that $x \in B$ if and only if $x \in A$ and P(x).

Pair: For any A and B, there is a C such that $x \in C$ if and only if $x = A$ or $x = B$.

Union: For any S, there is a U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

Power set: For any S, there is a P such that $X \in P$ if and only if $X \subseteq S$.

Infinity: There exists an inductive set.

Schema of Replacement: Let P(x, y) be a property such that for every x there is a unique y for which P(x, y). For every A there is a B such that for every $x \in A$ there is a $y \in B$ for which P(x, y).

Choice: Every system of sets has a choice function.

Peano

Peano Arithmetic is FOL with a single predicate for identity; functions for successor, addition, and multiplication; and the following axioms:

A1: $(x_1=x_2 \rightarrow (x_1=x_3 \rightarrow x_2=x_3))$
A2: $(x_1=x_2 \rightarrow x_1' = x_2')$
A3: $\neg 0=x_1'$
A4: $(x_1'=x_2' \rightarrow x_1=x_2)$
A5: $x_1+0 = x_1$
A6: $x_1+x_2'= (x_1 + x_2)'$
A7: $x_1 * 0 = 0$
A8: $x_1 * (x_2') = (x_1 * x_2) + x_1$
A9: For any wff **B(x)** of the theory, $(B(0) \rightarrow (\forall x(B(x) \rightarrow B(x')) \rightarrow \forall x(B(x))))$

We have also allowed substitution of identicals in Peano Arithmetic. In this exam you may also use commutitivity which says **t+r = r+t** and **t * r = r * t**.

You have been a great group! Have a good break and I hope to see you in the Spring. Email me if I can help you in any way next semester.