Selected Axiomatic Systems

Let ***PL*** be our propositional logic (Axioms L1-L3 and MP).
Let ***FOL*** be our first order logic (***PL***, Axioms L4 and L5, and Generalization).

Dedekind Style Axioms for Arithmetic

FOL with predicate F (identity) and functions f_1 (successor), f_2 (addition), and f_3 (multiplication); and the following axioms:

A1: $(x_1=x_2 \rightarrow (x_1=x_3 \rightarrow x_2=x_3))$
A2: $(x_1=x_2 \rightarrow x_1'=x_2')$
A3: $\neg 0=x_1'$
A4: $(x_1'=x_2' \rightarrow x_1=x_2)$
A5: $x_1+0 = x_1$
A6: $x_1+x_2' = (x_1 + x_2)'$
A7: $x_1 * 0 = 0$
A8: $x_1 * x_2' = (x_1 * x_2) + x_1$
A9: $(\Phi_{(0)} \text{-->} (\forall x(\Phi_{(x)} \text{-->} \Phi_{(x')}) \text{-->} \forall x(\Phi_{(x)})))$

Where "x=y" means Fxy, identity; "x'" means f_1x, the successor of x; "x+y" means f_2xy, addition; and "x*y" means f_3xy, multiplication.

We also add the rule, substitution of identicals (also sometimes called the "indiscernibility of identicals"). If x=y, then for any formula ϕ you can replace any x in ϕ with y, or any y in ϕ with x. Also, let's assume (this is derivable) commutivity of addition and of multiplication and of identity.

Primitive Recursion

FOL and the following functions, called the *initial functions:*

Zero: $\mathbf{Z(x) = 0}$ for all x
Successor: $\mathbf{N(x) = x+1}$ for all x
Projection: $\mathbf{U^n_i(x_1...x_n) = x_i}$ for all $x_1 \ldots x_n$

Rules are:

Substitution: $\mathbf{f(x_1...x_n) = g(h_1(x_1...x_n) \ldots h_m(x_1...x_n))}$
Recursion: $\mathbf{f(x_1...x_n, 0) = g(x_1...x_n)}$
$\mathbf{f(x_1...x_n, y+1) = h(x_1...x_n, y, f(x_1...x_n, y))}$

A function is ***primitive recursive*** if it can be obtained from finite instances of substitution or recursion starting with the initial functions.

Recursion

Axioms and rules for primitive recursion and the following rule:

μ-Operator: $\mathbf{f(x_1...x_n) = \mu y(g(x_1...x_n, y) = 0)}$

where there is at least one y such that $g(x_1...x_n, y) = 0$ and $\mu y(g(x_1...x_n, y) = 0)$ is the least y such that $g(x_1...x_n, y) = 0$.

A function is ***recursive*** if it can be obtained from finite instances of substitution or recursion or the μ-Operator, starting with the initial functions.

Standard Modal Propositional Logics

M **(aka T)**

PL and:

M1: $(\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi))$

M2: $(\Box\phi \to \phi)$

Brouwer

PL and ***M*** and:

M3: $(\phi \to \Box\Diamond\phi)$

S4

PL and ***M*** and:

M4: $(\Box\phi \to \Box\Box\phi)$

Note corollary theorems of **S4**:

$(\Box\phi \leftrightarrow \Box\Box\phi)$

$(\Diamond\phi \leftrightarrow \Diamond\Diamond\phi)$

S5

PL and ***M*** and:

M5: $(\Diamond\phi \to \Box\Diamond\phi)$

Note corollary theorems of **S5**:

$(\Box\phi \leftrightarrow \Diamond\Box\phi)$

$(\Diamond\phi \leftrightarrow \Box\Diamond\phi)$