A ZF-style axiom system

ZF is the **Z**ermelo-**F**rankel set theory. It is not uncommon to state the axioms in English, since they are meant to translate across various versions of the First Order Logic, or to a natural deduction system version of the FOL. We add here typical formal statements.

Existence: There exists a set that has no elements.

$$\exists x \forall y (y \notin x)$$

Extensionality: Every element of x is an element of y, and every element of y is an element of x, if and only if x=y.

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$$

Schema of comprehension: Let $\phi(z)$ be a property of z. For any x, there is a y such that $z \in y$ if and only if $z \in x$ and $\phi(z)$.

$$\forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \phi(z)))$$

Pair: For any x and y, there is a z such that $u \in z$ if and only if $u = x$ or $u = y$.

$$\forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u=x \vee u=y))$$

Union: For any x, there is a y such that $z \in y$ if and only if for some v, $z \in v$ and $v \in x$.

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists v (z \in v \wedge v \in x))$$

Power set: For any x, there is a y such that $z \in y$ if and only if $z \subseteq x$.

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x)$$

Infinity: There exists an inductive set.

$$\exists x (\emptyset \in x \wedge \forall u (u \in x \rightarrow u \cup \{u\} \in x))$$

You get ZFC by adding:

Schema of Replacement: Let P(x, y) be a property such that for every x there is a unique y for which P(x, y). For every A there is a B such that for every $x \in A$ there is a $y \in B$ for which P(x, y).

Choice: Every system of sets has a choice function.